# Elastic and Flow Mechanics for Membrane Spargers

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## Introduction

The recent development of elastic membrane spargers (Rice et al., 1981; Rice and Lakhani, 1983; Rice and Howell, 1984) has led to fundamental questions regarding the basic principles to explain the reported enhanced behavior, which includes large increases in transfer coefficients, increased voidage, and extended operational range (self-regulation). The proper first step in the effort to provide an explanation is to develop a relationship to show the connection between elastic and flow mechanics when a punctured membrane is subjected to pressure from below. From this, a theory emerges to predict the way flow through the membrane changes as a function of pressure, extension ratio, and elementary transport propeties such as shear modulus and gas viscosity.

## Elastic Mechanics

The mechanical system is composed of an elastic membrane stretched over a cylindrical drumhead and clamped around the outer perimeter, as shown in Figure 1a. The membrane is inflated by applying pressure from below, and in principle the shape of the inflating membrane can be computed. The stress-strain relations are usually expressed in terms of tension-extension ratio relations. Thus, Figure 1b shows a cross section of a circular element in the expanded membrane, and Figure 1c shows the element in relation to the unexpanded (flat) state. In the analysis to follow, we shall take the membrane to be thin; hence body forces are negligible. In the equilibrium state, $T_1$ is the meridian tension and $T_2$ is the hoop tension, so that (Williams, 1980):

$$T_1 = P_1 r/(2 \sin \alpha) \tag{1}$$

$$T_2 = \frac{d}{dr}(rT_1) \tag{2}$$

The membrane deforms in three directions. Therefore, the extension ratios, $\lambda_i$, are defined as the distorted length divided by the initial, undistorted length. Referring to Figure 1c, one defines:

$$\lambda_1 = \frac{\Delta S}{\Delta r_o} = \frac{\Delta r}{\Delta r_o}\frac{1}{\cos \alpha} \tag{3}$$

$$\lambda_2 = r/r_o \tag{4}$$

$$\lambda_3 = b/b_o \tag{5}$$

and for incompressible solids, a material balance requires

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{6}$$

The stress-strain relations are determined from stored energy in terms of the so-called strain invariants (Rivlin and Saunders, 1951). Thus, in terms of the work of deformation per unit volume of material (stored free energy), $W_v$, the tension components can be expressed as:

$$T_1 = 2\lambda_3 b_o(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W_v}{\partial I_1} + \lambda_2^2 \frac{\partial W_v}{\partial I_2}\right) \tag{7}$$

$$T_2 = 2\lambda_3 b_o(\lambda_2^2 - \lambda_3^3)\left(\frac{\partial W_v}{\partial I_1} + \lambda_1^2 \frac{\partial W_v}{\partial I_2}\right) \tag{8}$$

The strain invariants for an incompressible solid can be defined as:

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{9}$$

$$I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \tag{10}$$

When extension ratios are small, the classical statistical theory of polymer elasticity (Treloar, 1975) suggests:

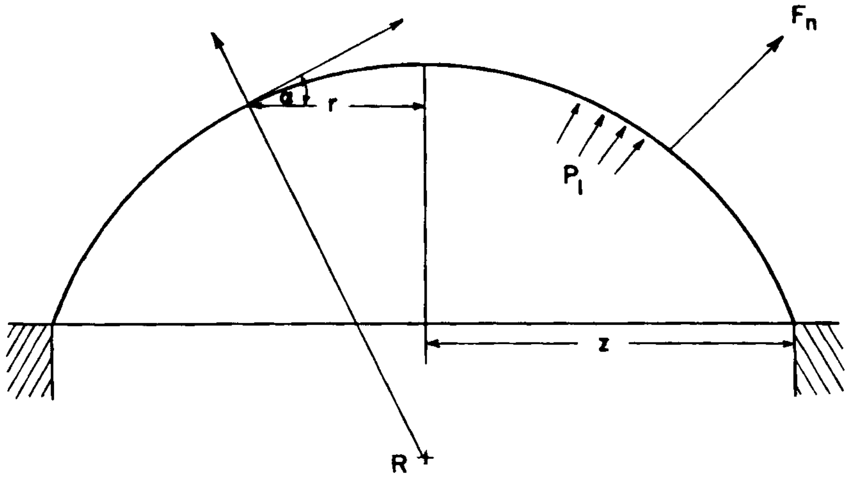


**Figure 1a. Expanded membrane: overall shape and dimensions.**

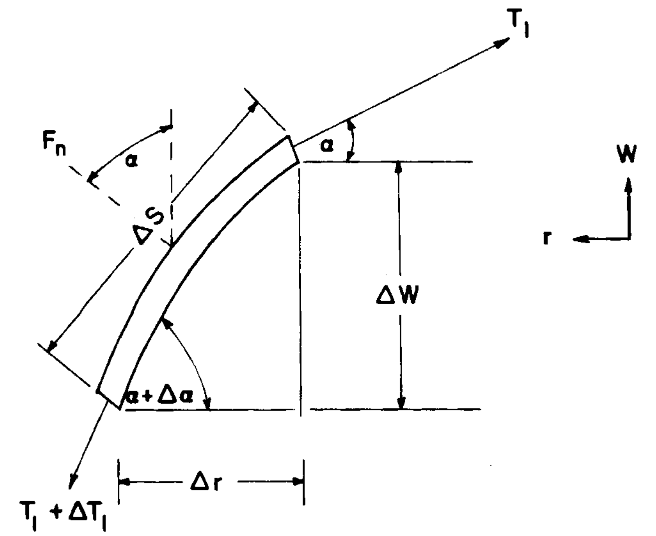


**Figure 1b. Expanded membrane: schematic of elemental force balance.**

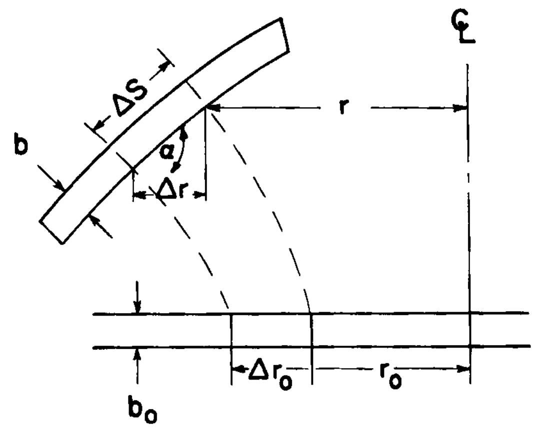


**Figure 1c. Expanded membrane: extension ratio definitions.**

$$W_v = \frac{1}{2} G(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \tag{11}$$

A final equation is needed to describe extension as a function of position along the surface of the expanded membrane. To get this, we rearrange Eq. 4 to read:

$$r_o = \frac{r}{\lambda_2(r)}$$

Differentiating this and substituting into Eq. 3 yields:

$$\frac{d\lambda_2}{dr} = \frac{\lambda_2}{r}\left(1 - \frac{\lambda_2}{\lambda_1 \cos\alpha}\right) \tag{12}$$

Thus, we have a system of equations comprised of Eqs. 1, 2, 7, 8, and 12 to solve for $T_1$, $T_2$, $\lambda_1$, $\lambda_2$, and $\alpha$ ($\lambda_3$ being expressible using Eq. 6). These nonlinear sets have been solved numerically by Rivlin and Saunders (1951) and Yu and Valanis (1968). The shape for small extension ratios is very nearly that of a spherical cap, and deviates slightly from this shape as the secured edge is approached.

At the pole, by symmetry we have $T_1 = T_2 = T$, and $\lambda_1 = \lambda_2 = \lambda$, hence $\lambda_3 = 1/\lambda^2$. These relations are also valid when the membrane takes the shape of a spherical cap, which will be assumed henceforth.

Following Williams (1980) for a spherical cap membrane, we have:

$$r = R \sin\alpha \tag{13}$$

$$dr = R \cos\alpha d\alpha \tag{14}$$

where $R$ is the projected radius of a sphere. Since $\lambda_1 = \lambda_2 = \lambda$, then $\lambda_3 = 1/\lambda^2$; hence we equate the relations in Eqs. 3 and 4 to get

$$\frac{r}{r_o} = \frac{1}{\cos\alpha}\frac{dr}{dr_o} \tag{15}$$

Combining this with Eqs. 13 and 14, and integrating yields:

$$r_o/B = \frac{\sin\alpha}{1+\cos\alpha} = \frac{r}{R+\sqrt{R^2-r^2}} \tag{16}$$

The arbitrary constant of integration, $B$, can be evaluated using conditions at the fixed edge of the membrane, where $r_o = Z$, and since it can be taken that there is no extension at the edge, then $r = Z$; thus $r_o = r$ so that

$$B = R + \sqrt{R^2 - Z^2} \tag{17}$$

Inserting $B$ into Eq. 22 gives a general expression for $\lambda$ as a function of distance from the pole:

$$\lambda = \lambda_1 = \lambda_2 = \frac{r}{r_o} = (R + \sqrt{R^2 - r^2})/(R + \sqrt{R^2 - Z^2}) \tag{18}$$

It is convenient from an experimental point of view to express $\lambda$ in terms of deflection from a flat initial shape. So, taking $W_o$ as the deflection at the pole and $W(r)$ as the deflection for any value of $r$, we can express the membrane surface area two ways:

$$A = \pi Z^2 + \pi W_o^2 = 2\pi R W_o \tag{19}$$

hence the radius of curvature is:

$$R = (Z^2 + W_o^2)/2W_o \tag{20}$$

The deflection $W(r)$ can then be expressed:

$$W(r) = W_o - R + \sqrt{R^2 - r^2} \tag{21}$$

which leads to:

$$\lambda(r) = 1 + \frac{W(r)W_o}{Z^2} \tag{22}$$

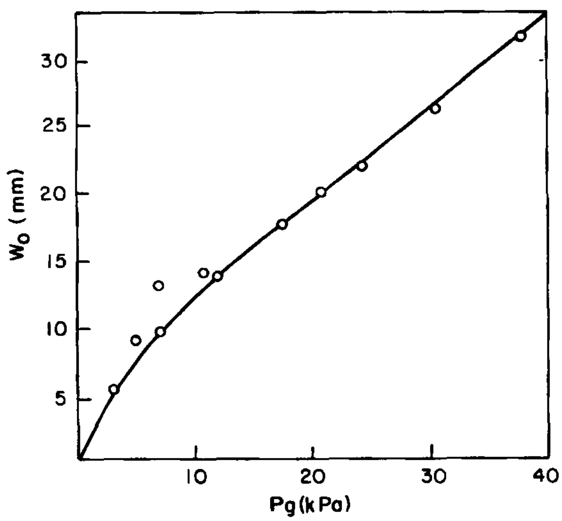


**Figure 2a. Deflection at the pole measurements: latex (2.03 mm thick).**

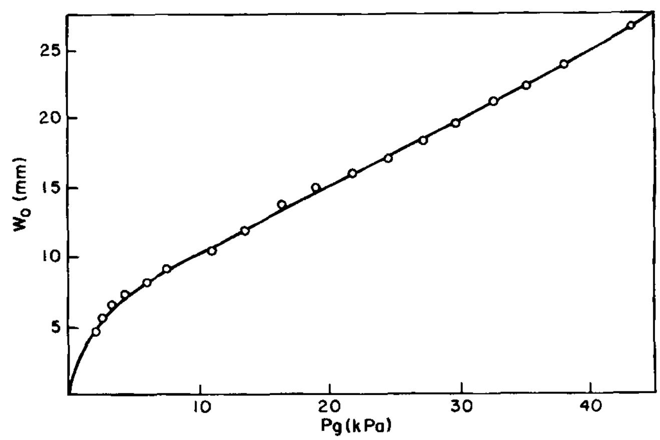


**Figure 2b. Deflection at the pole measurements: EPDM (1.68 mm thick).**

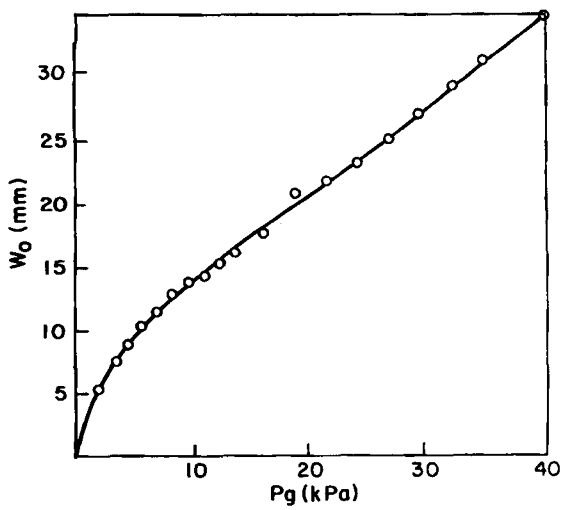


**Figure 2c. Deflection at the pole measurements: neoprene (1.303 mm thick).**

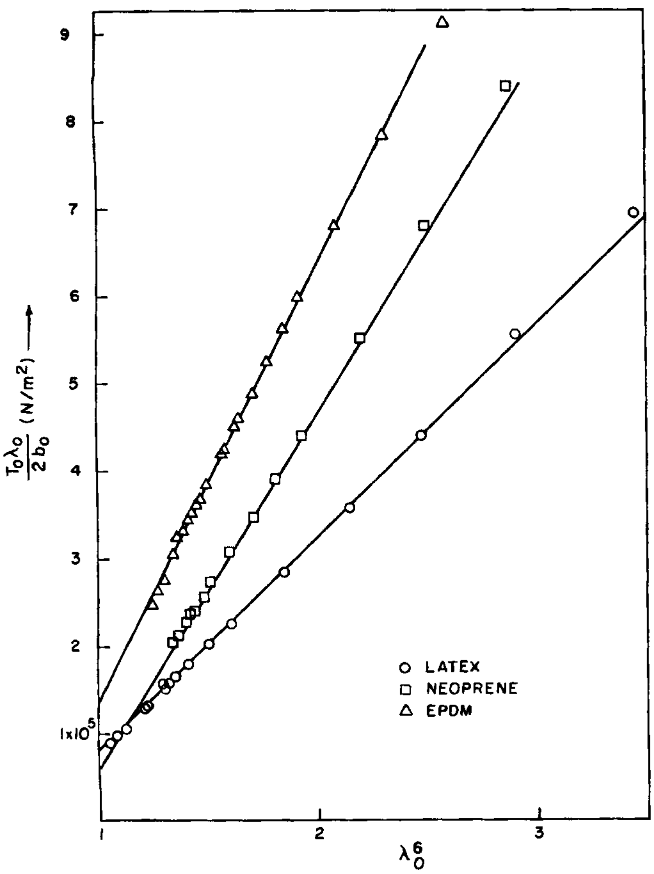


**Figure 3. Test of Eq. 27 to verify statistical energy model.**

and at the pole where $r = o$:

$$\lambda_o = 1 + W_o^2/Z^2 \tag{23}$$

Thus, the extension ratio at the pole can be computed in terms of measured deflection at the pole.

Since $\lambda_1 = \lambda_2$ under the present conditions, then $T_1 = T_2$; hence either Eq. 7 or Eq. 8 can be written (since $\lambda_3 = 1/\lambda^2$):

$$T(r) = 2b_o(1 - 1/\lambda^6)\left(\frac{\partial W_v}{\partial I_1} + \lambda^2\frac{\partial W_v}{\partial I_2}\right) \tag{24}$$

Applying the statistical theory for energy storage (Eq. 11) gives the work function:

$$W_v = \frac{1}{2}G(2\lambda^2 + 1/\lambda^4 - 3) \tag{25}$$

Differentiating this, using the definition of $I_1(\lambda)$ and $I_2(\lambda)$ given by Eqs. 9 and 10, yields:

$$T(r) = 2b_oG(1 - 1/\lambda^6) \tag{26}$$

and at the pole we finally have:

$$T_o = 2b_oG(1 - 1/\lambda_o^6) \tag{27}$$

Thus, elementary data relating deflection, $W_o$, to pressure, $P_1$, such as shown in Figure 2 (Howell, 1984), can be used to find shear modulus as follows. One computes $R$ from Eq. 20, then $\lambda_o$ is obtained from Eq. 23, and finally $T_o$ is calculated using the well-known relation $P_1 = 2T_o/R$. These values are used in Eq. 27 by plotting $T_o\lambda_o^6/2b_o$ vs. $\lambda_o^6$, from which the elastic constant, $G$, can be estimated by taking slopes. Rice and Lakhani (1983) compared actual membrane shapes (using a flexible curve) with the spherical cap model from which Eq. 27 was obtained and found very good agreement for a latex membrane.

Figure 3 shows our apparently new method to test the classical theory based on membrane deflections at the pole. The three elastic materials (which are hardy candidates for process applications) tested were neoprene, EPDM (ethylene propylene diene monomer), and latex, all of which were clamped on a 4" (0.102 m) drumhead. Experimental membrane thicknesses are indicated in Table 1. A remarkably uniform straight line resulted, which thereby justifies the use of the classical theory. Table 1 shows the value of shear modulus $G$ obtained from the slopes of the curves in Figure 3.

## Bubble Formation at Punctured Membranes

The elastic theory can be applied directly in the calculation of hole or puncture size for applications to membrane sparger design. Thus Rivlin and Thomas (1951) confirmed that the radius of a hole in a sheet undergoing uniform extension ($T_1 = T_2$) is:

$$r_H = r_{H_i}\lambda \tag{28}$$

**Table 1. Comparative Values of Shear Modulus *G***

| Material | Dia. m | Values Deduced from Slopes in Figure 3: $G$ N/m² | $N^*$ | $b_o$ mm | Reference |
|---|---|---|---|---|---|
| Neoprene | 0.102 | $4.1 \times 10^5$ | $1 \times 10^{26}$ | 1.303 | This work |
| EPDM | 0.102 | $4.9 \times 10^5$ | $1.2 \times 10^{26}$ | 1.68 | This work |
| Latex | 0.102 | $2.5 \times 10^5$ | $0.6 \times 10^{26}$ | 2.03 | This work |
| | | Literature Values | | | |
| Natural rubber | — | $3.9 \times 10^5$ | — | — | Treolar (1975) |
| Gum stock | — | $3.33 \times 10^5$ | — | — | Mooney (1940) |
| Tread stock | — | $6.65 \times 10^5$ | — | — | Mooney (1940) |
| Natural rubber | — | $3.2$ to $3.4 \times 10^5$ | — | — | Rivlin & Saunders (1951) |

*$N$ = number of polymer chains in a network per m³, computed from $G = NkT_A$.

Because the pressure drop is quite large for membrane spargers, constant flow conditions nearly always prevail during bubble formation. It is our purpose here to show that useful information is directly obtainable by *in situ* measurements of a bubbling sparger. For simplicity, a single, centered hole will be analyzed. We start with the Bernoulli relationship, which is modified to get the well-known orifice equation:

$$G_v = A_H C_o \sqrt{\frac{2}{\rho_g} \Delta p} \tag{29}$$

For the small punctures used in the present work, flow is almost always laminar ($Re < 100$) so the orifice coefficient varies as (Perry and Chilton, 1973)

$$C_o = K_o \sqrt{Re} \tag{30}$$

and the hole area is

$$A_H = \pi \lambda_o^2 r_{H_i}^2 \tag{31}$$

The pressure change is given by (Rice and Lakhani, 1983):

$$\Delta p = P_g - \rho g h - \Delta p_c = P_1 - \Delta p_c \tag{32}$$

where the critical pressure, $\Delta p_c$, is a measured quantity and denotes the pressure necessary to form the first detectable bubble.

Rearranging the above leads to:

$$(G_v/\Delta p)^{1/3} = \lambda_o r_{H_i} \left( \frac{4\pi K_o^2}{\mu_g} \right)^{1/3} \tag{33}$$

which suggests a linear relationship with extension ratio at the pole. Figure 4 shows experimental tests of this theory for a latex membrane. At the intercept where $\lambda_o \rightarrow 1$, one can estimate the original puncture size. Happel and Brenner (1965) provide an exact solution for laminar flow through a circular aperture, and their results suggest $K_o = 1/\sqrt{12\pi}$. A simple linear relationship between laminar flow and pressure drop (as for rigid holes) does not occur for elastic holes.

## Dimensionless Pressure-Flow Relation

Now that the elastic and fluid mechanical models have been separately verified, one can use the estimated values of shear modulus $G$ and initial puncture size to formulate two unique dimensionless groups that lead to a simple expression relating flow and pressure through an elastic hole.

We first express tension in terms of pressure using (Rivlin and Saunders, 1951):

$$T_o = P_1 R/2 \tag{34}$$

Next, we express radius of curvature $R$ in terms of dimensionless deflection using Eq. 20:

$$R = [1 + (W_o/Z)^2] \cdot Z/(2W_o/Z) \tag{35}$$

Placing this into Eq. 34 and inserting Eq. 23 finally gives an expression for tension at the pole:

$$T_o = \frac{P_1 Z}{4} \left( \frac{\lambda_o}{\sqrt{\lambda_o - 1}} \right) \tag{36}$$

Inserting this in Eq. 27 leads to the dimensionless expression:

$$\left( \frac{P_1 Z}{8 b_o G} \right) = \left( \frac{\sqrt{\lambda_o - 1}}{\lambda_o} \right) \left( \frac{\lambda_o^6 - 1}{\lambda_o^6} \right) = N_p \tag{37}$$

where we define the dimensionless pressure number, $N_p$. We can express Eq. 33 in terms of extension ratio:

$$[3\mu_g G_v/(\Delta p r_{H_i}^3)]^{1/3} = \lambda_o = N_f^{1/3} \tag{38}$$

where we define the flow dimensionless group, $N_f$.

Finally, we combine the last two results to get a general result for laminar flow through elastic holes:

$$N_p = \frac{(N_f^{1/3} - 1)^{1/2}}{N_f^{1/3}} \left( \frac{N_f^2 - 1}{N_f^2} \right) \tag{39}$$

The general theory is tested in Figure 5, using data taken from Figures 3 and 4. The assumption of a spherical cap mem-

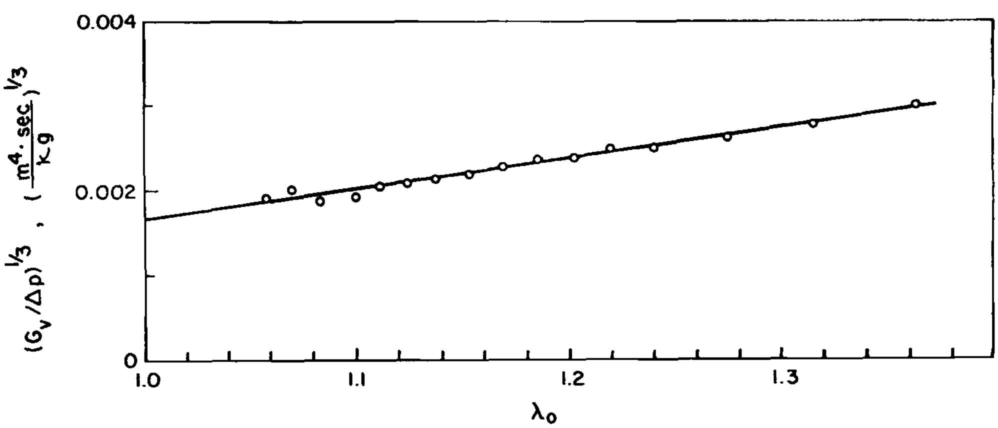


**Figure 4. Test of Eq. 33 for linearity of extension ratio.**

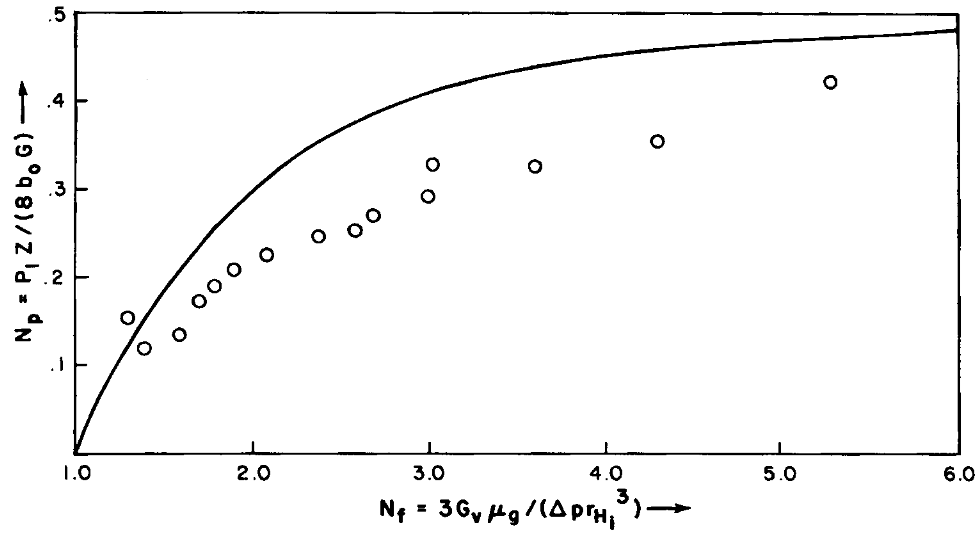


**Figure 5. Test of general theory, Eq. 39.**
Points represent experimental data from Fig. 3.

brane sustaining laminar hole flow seems to be supported by experimental observations.

## Acknowledgment

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## Notation

$A_H$ = hole or puncture area for flow
$A_{H_c}$ = hole or puncture area at lowest flow obtainable (critical area)
$b$ = thickness of expanded elastomer
$b_o$ = initial (undeflected) thickness of elastomer sheet
$C_o$ = orifice coefficient
$E$ = Young's modulus
$G_v$ = gas flow rate
$G$ = shear modulus
$h$ = liquid height above hole
$I_j$ = $j$th strain invariant
$k$ = Boltzman's constant
$K_o$ = dimensionless coefficient for laminar orifice, $1/\sqrt{12\pi}$
$N_f$ = dimensionless flow number, $3G_v\mu_g/\Delta p r_{H_i}^3$
$N_p$ = dimensionless pressure number, $P_1 Z/8b_o G$
$P_g$ = gage pressure
$P_1$ = $P_g - \rho g h$
$P_o$ = elastic pressure (minimum gage pressure to attain flow in air-only situation)
$\Delta p$ = pressure change for flow through aperture
$\Delta p_c$ = critical pressure
$r_o$ = radial coordinate for flat membrane
$r$ = radial coordinate for expanded membrane
$r_H$ = effective hole radius
$r_{H_c}$ = critical hole radius corresponding to $\Delta p_c$
$r_{H_i}$ = radius of pseudoinitial hole
$R$ = radius for spherical cap
$Re$ = Reynolds number, $u_H d_H \rho_g/\mu_g$
$S$ = arc length on sparger surface
$T_A$ = absolute temperature
$T_j$ = $j$th component of tension
$T_o$ = uniform tension at pole
$u_H$ = linear velocity through hole
$W_v$ = stored energy function
$W(r)$ = vertical deflection of membrane
$W_o$ = vertical deflection of membrane at pole
$Z$ = radius of drumhead

### *Greek letters*

$\xi$ = Poisson's ratio
$\lambda_j$ = $j$th component of extension ratio
$\lambda_o$ = extension ratio at the pole
$\mu_g$ = gas viscosity
$\rho_g$ = gas density
$\rho$ = liquid density
$\sigma$ = surface tension of liquid

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